Causal description of heat and mass transfer

2004 J. Phys. A: Math. Gen. 37 3053

(http://iopscience.iop.org/0305-4470/37/8/015)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 3053–3069 PII: S0305-4470(04)69489-3

Causal description of heat and mass transfer

M Belevich

St Petersburg Branch, Institute of Oceanology, Russian Academy of Sciences, 1st Line 30, St Petersburg 199053, Russia

E-mail: mb@ioras.nw.ru

Received 25 September 2003
Published 11 February 2004
Online at stacks.iop.org/JPhysA/37/3053 (DOI: 10.1088/0305-4470/37/8/015)

Abstract

Methods of causal description for transport phenomena are developed. The approach applied is based on introduction of the upper limit velocity which is interpreted as a speed of the signal used for observation. Differences in the heat and mass transfer are considered. Corresponding evolution equations which satisfy the causality principle are derived.

PACS numbers: 51.20.+d, 66.10.Cb, 66.20.+d, 66.60.+a, 66.90.+r

1. Introduction

Apart from deformation and transporting of finite volumes the hydrodynamic evolution of fluids includes the so-called transport phenomena caused by the general tendency to equilibrium which does not depend on a particular matter (the classical viewpoint on these phenomena may be found in [1]). If some parameters in the initial state are distributed inhomogeneously, an exchange towards equilibrium takes place. The three kinds of this phenomenon include matter (mass), energy (heat) and momentum transfer. The classical mass–heat–momentum transfer equations are widely used and are applicable to a variety of problems. However, there are at least two reasons to seek out non-classical approaches to describe these phenomena.

First, there are certain cases, in practice, when solutions of the corresponding parabolic equations fail to predict and/or approximate the experimental data. This mainly concerns the experimental studies of finite-speed propagation of the thermal signals at low temperatures [2–5] as well as laser and microwave heating with extremely short duration or high frequencies (see, e.g., [6, 7]).

Second, the classical description is unacceptable due to a fundamental reason: it contradicts the causality principle. Classical theory models the transport phenomena using the parabolic equations which are known to have an exponential fundamental solution (see, e.g., [8]). This feature may be physically interpreted as a possibility of propagation of disturbances with infinite velocities. In other words such a model permits an effect to be synchronous

with its cause. It is possible to justify the practical use of the classical approach in various computations considering the rate of attenuation of the infinitely fast waves (see [9, 10]). However, the presence itself of the infinite velocities testifies to ill-health of the theory. As long as the causality principle is the basis of natural science, any theory which does not satisfy it may be regarded as a useful computational tool, but not an adequate description of the phenomenon.

According to the common viewpoint the transport phenomena differ in transferring quantity only (mass, heat, momentum), whereas the mechanism in all cases is the same and is understood as transfer due to the chaotic movements of the molecules and other microscopic particles. This is the reason why the derivation of hyperbolic transport equations is often started from the Boltzmann equation [11, 12]. Returning to considerations of the particle dynamics after the continuity hypothesis is already postulated may be helpful in the search for ideas. However, after the idea is found, it is desirable to derive corresponding equations in the framework of the continuum being based on integral relations such as conservation laws. This is especially important since only integral relations are verifiable in practice, whereas differential relations are actually non-verifiable by definition.

A drawback of the particle dynamics approach seems to be in a loss of fundamental differences in the transport of various quantities. The point is that while viscosity and conductivity are two aspects of energy transfer, diffusion is another matter. First of all, in the framework of continuum mechanics it is not connected with chaotic movements of the points of the fluid since all of these movements are already described using viscosity/conductivity by definition. Second, diffusion is the transport of mass, and hence is a current. The fact that all of the transport phenomena are described by the same differential equation, within the notation, does not identify their actual similarity. The motives which cause the description of the flow of diffusing matter using the diffusion equation, and not the equations of motion, we try to analyse in what follows below.

As has already been said, hyperbolic equations for the transfer problems have been suggested in a number of publications. Besides the works cited above, some causal theories for heat and mass transfer have been developed using the approach known as extended irreversible thermodynamics. A review of these theories may be found in [13–16] and a brief critical analysis is presented in [17]. A non-Fickian diffusion in the framework of extended irreversible thermodynamics is considered in [18]. Despite this activity, an accepted theory has not yet been proposed and development of a causal description is still a topical problem. The current paper is another attempt in this field.

We use the unified approach suggested recently in [19] and successfully applied to the description of the dissipative fluid flow. It is based on the obvious fact that an adequate model may be constructed only using information about the phenomenon which reaches the observer with some finite velocity (luminal, sound, etc). Introduction of finite information (or signal) speed in a model of continuum prevents the occurrence of velocities which exceed the signal velocity, otherwise they are either unobservable (the signal is unable to catch up with the object), or (the object moves towards the signal) an apparent velocity smaller than that of the signal is observed. Taking into account the observer and explicit consideration of the finiteness of the signal velocity made it possible to derive a symmetric hyperbolic system of hydrothermodynamic equations with a stable equilibrium state. This model of the fluid is based on the general integral conservation laws and does not require any phenomenological parameters in addition to the classical ones.

This paper is organized as follows. Section 2 briefly describes the causal fluid model suggested in [19]. In the third section we discuss the averaging procedure which differs to some extent from the one usually applied. The fourth section deals with the viscous transport

of momentum and heat conduction. Both of these processes are closely connected and essentially describe parametrically the whole spectrum of the small-scale movements which we are unable to describe explicitly. Hyperbolic balance equations for kinetic and internal energy, the momentum balance equation and the heat equation are derived here. Section 5 is devoted to the diffusion phenomenon. Contrary to viscosity and conductivity, diffusion is regarded as a transport of mass and thus as a current. However, the diffusion of a component of a mixture is a current inside the current of the mixture as a whole. As a result, any deformation lacks its bijectivity and this gives rise to certain difficulties in the description. The hyperbolic diffusion equation derived in this section allows one to use the hyperbolic fluid dynamic equations of [19] in the case of mixtures as well. In the sixth section we discuss the results obtained.

It is necessary to emphasize that all considerations are undertaken in the framework of the continuum. The continuity hypothesis is assumed to hold, and the set of molecules is substituted for the continuum of points which is described by the smooth functions.

2. Causal fluid model

Following [20] we shall think of a fluid body $\mathcal{B}$ as a set in a topological space. Each point of the body is associated with its *worldline* in the *space of events* $\mathcal{W}$ or the four-dimensional spacetime continuum. This space $\mathcal{W}$ may be thought of as a congruence of the worldlines. The worldlines of the points of the body in total form the *worldtube* of the body $\mathcal{B}$ which is considered as a four-dimensional manifold $\mathcal{B}^4$ in the space $\mathcal{W}$.

2.1. Time and coordinate systems

Time is introduced via parametrization of the worldlines. Each worldline is equipped with a real parameter t and all of these parameters are called *time*. Since the parametrization is arbitrary additional requirements may be used to choose the most appropriate one. Our requirement is connected with synchronization of times of different worldlines. We choose one special worldline with arbitrary parametrization which will be called the worldline of the *observer*. All other parameters will be synchronized with the observer's time.

It is convenient to choose a synchronization procedure such that it may be interpreted in terms of the velocity of a signal which is used for soundings of the medium and carrying the information back to the observer (see, e.g., [21]). Let the information about the fluid motion reach the observer with a finite signal velocity. We call the *signal* a monochrome radiation which weakly interacts with the fluid in study. This means that the signal propagating in the fluid does not change its physical properties. The phase speed of the signal will be denoted as c.

As a result only one time is left, the time of the observer since all other times are synchronized with it. Taking into account the existence of a preferential worldline of the observer, the coordinate systems will be defined such that this worldline coincides with the axis connected with time. The spacetime continuum is assumed to be homogeneous, thus the origin of the coordinate system may be chosen at an arbitrary point of the observer's worldline.

2.1.1. Euler coordinates. Consider a space of events $\mathcal{W}$ as a direct product of the observer's worldline and three-dimensional space of synchronous (with respect to the observer's time) events. We map the space of synchronous events onto $\mathbb{R}^3$ and the observer's worldline onto the space of imaginary numbers which will be denoted as $\mathrm{i}\mathbb{R}^1$. This trick allows unified treatment of temporal and spatial variables. Now the new map $\phi^t : \mathcal{W} \to \mathrm{i}\mathbb{R}^1 \times \mathbb{R}^3$ which is called the

observer's frame of reference may be considered. This map equips each point $P \in \mathcal{W}$ with four numbers $x_P = (x^0, x^1, x^2, x^3)$, the coordinates of the point. The worldline of the body point is defined by four functions $x^0(t), x^1(t), x^2(t), x^3(t)$ which give for each time value the Euler coordinates of the point of the body. The observer's worldline is the line $(x^0(t), 0, 0, 0)$ by definition. It is assumed that $\mathrm{d}x^0 = \mathrm{i}c\mathrm{d}t$, where $\mathrm{i} = \sqrt{-1}$ and c is the phase velocity of the signal which is used for observations and measurement of the fluid motion. The tangent 4-velocity vector $\vec{v}_P = \mathrm{d}_t x_P$ is defined at each point of any of the worldlines. Thus, the vector field V is defined on the manifold $\mathcal{B}^4$. We further define at each point of $\mathcal{W}$ the coordinate basis of tangent vectors $\{\vec{e}_\alpha\}_{\alpha=0}^3$, $\vec{e}_\alpha = \partial_{x^\alpha}$. Then, $\vec{v} = v^\alpha \vec{e}_\alpha$ (hereafter the index summation convention is used).

2.1.2. Lagrangian coordinates. Let the vector field V be smooth. At each point of $\mathcal{B}^4$ the tangent vector $\vec{v} \neq 0$ due to $v^0 = \mathrm{i}c \neq 0$. Under these conditions the following theorem holds (see, e.g., [22]): *for small enough vicinity of the arbitrary non-singular point of* $\mathcal{B}^4$ *it is possible to choose a system of coordinates* (X^0, X^1, X^2, X^3) *such that the vector field* V *on* $\mathcal{B}^4$ *is constant* $(\partial_{X^0}, 0, 0, 0)$. Classical fluid mechanics calls the numbers X^1, X^2, X^3 the Lagrangian labels or coordinates. By analogy, the four numbers (X^0, X^1, X^2, X^3) where $\mathrm{d}X^0 = \mathrm{i}c\mathrm{d}t$ will here be called the *Lagrangian coordinates* of the point of the body. Since the spatial components of the Lagrangian coordinates do not depend on time, the worldline of each point is a line parallel to the worldline of the observer $(X^0, 0, 0, 0)$, i.e. the X^0 axis.

So, the point $\mathcal{P}$ of the body $\mathcal{B}$ located at time t_P at the point P of the space of events $\mathcal{W}$ is described both by its Lagrangian coordinates $X_P = \left(X_P^0, X_P^1, X_P^2, X_P^3\right)$ and by its Euler coordinates $x_P = (x^0(t_P), x^1(t_P), x^2(t_P), x^3(t_P))$.

2.2. Measures and mass conservation law

Synchronous cut $\mathcal{B}_t$ of the worldtube will be called the *configuration* of the body at time t. It is possible to introduce measures on $\mathcal{B}_t$ with the usual interpretation of mass, volume and energy of the body. Let $m(\mathcal{B}_t)$ and $\omega(\mathcal{B}_t)$ be such measures which we shall call the *mass* and *volume* of $\mathcal{B}_t$, respectively. Mass m is assumed to be absolutely continuous with respect to the volume ω, i.e. $\omega(\mathcal{P}) = 0 \Leftrightarrow m(\mathcal{P}) = 0$ for any configuration of the arbitrary body $\mathcal{P}$. In this case, the connection between both measures m and ω is established by the Radon–Nikodym theorem [23], according to which there exists a unique function ρ defined on $\mathcal{B}_t$ such that

$$m(\mathcal{B}_t) = \int_{\omega(\mathcal{B}_t)} \rho \, \mathrm{d}\omega. \tag{1}$$

The function ρ is called the derivative of measure m with respect to measure ω or the *mass density*, since m is the mass. The mass conservation law

$$\mathrm{d}_t m(\mathcal{B}_t) = 0 \tag{2}$$

is postulated.

Apart from the mass m the volume $\omega(\mathcal{B}_t)$ may vary with time. Let Ω denote the volume of worldtube cut at the reference time t_0. Then, for arbitrarily located configuration $\mathcal{B}_t$ we may write $\mathrm{d}\omega(\mathcal{B}_t) = J\mathrm{d}\Omega$. Function J describes a change of the volume $\mathrm{d}\omega(\mathcal{B}_t)$ with time relative to the reference volume $\mathrm{d}\Omega$. The connection between the two measures m and Ω is similar to relation (1)

$$m(\mathcal{B}_t) = \int_\Omega \rho J \, \mathrm{d}\Omega. \tag{3}$$

Since Ω does not depend on time, equation (2) leads to the conservation law

$$\int_{\Omega} \mathrm{d}_t(\rho J)\,\mathrm{d}\Omega = 0.$$

On the assumption that the integrand is a continuous function and $\mathcal{B}_t$ is arbitrary we obtain

$$\mathrm{d}_t(\rho J) = 0. \tag{4}$$

When ρ is interpreted as a mass density, equation (4) is a four-dimensional continuity equation. We shall consider non-singular motion, i.e. such that there exists a 1–1 coordinate transformation $X_P \mapsto x_P$. From this coordinate point of view relations (1) and (3) may be interpreted as the mass of the worldtube cut $\mathcal{B}_t$ expressed in terms of the Euler and Lagrangian coordinates, respectively. The time-dependent function J in formula (3) is nothing but the Jacobian of the coordinate transformation $J = \det(\partial_{X^\alpha} x^\beta)$. Using the coordinate systems introduced, equation (4) may be written as follows:

$$\operatorname{div}(\rho\vec{v}) = 0. \tag{5}$$

2.3. Metric tensor and kinetic energy conservation law

The interpretation given to equation (4) is not unique. Another possible interpretation requires consideration of a tangent vector field V induced on the worldtube $\mathcal{B}^4$ by the parametrization chosen. Consider the quantity $k = \frac{1}{2}\rho|\vec{v}|^2$ which will be called *kinetic energy density*. To express this quantity in terms of the velocity components, we need the introduction of a metric tensor, i.e. a real linear symmetric two-argument function $\mathsf{g}(\vec{v}, \vec{w})$ defined on the vectors. Using the standard notation, the absolute values of the velocity vector and kinetic energy density are then written as follows:

$$|\vec{v}|^2 = \mathsf{g}_{\alpha\beta} v^\alpha v^\beta \qquad k = \tfrac{1}{2}\rho \mathsf{g}_{\alpha\beta} v^\alpha v^\beta. \tag{6}$$

Components of the metric tensor depend on the choice of the basis vectors and in the case of orthogonal basis, tensor g is diagonal

$$|\vec{v}|^2 = \sum_{\alpha=0}^{3} \mathsf{g}_{\alpha\alpha} v^\alpha v^\alpha.$$

It is convenient to define the metric tensor such that the length of the velocity vector $\vec{v}$ tangent to the worldline is constant, although arbitrary. For simplicity we choose $|\vec{v}| = 1$. If we set $\mathsf{g}_{\alpha\alpha} = g_0$ for all α, then

$$g_0 = \frac{1}{\sum_\alpha v^\alpha v^\alpha} \tag{7}$$

and the metric tensor is proportional to the identity tensor I, i.e. $\mathsf{g} = g_0\mathsf{I}$.

In this case the kinetic energy density is equal numerically to half the value of the mass density and equation (4) may be treated as the kinetic energy conservation law

$$\mathrm{d}_t(kJ) = 0. \tag{8}$$

Thus the other meaning of the measure $m(\mathcal{B}_t)$ is the *kinetic energy* of the worldtube cut

$$K(\mathcal{B}_t) = \int_{\Omega} kJ\,\mathrm{d}\omega = \int_{\omega(\mathcal{B}_t)} k\,\mathrm{d}\omega.$$

By analogy we write (8) as

$$\operatorname{div}(k\vec{v}) = 0. \tag{9}$$

3. Averaging and total energy conservation law

3.1. Averaging

Since fluid mechanics is based on the continuity hypothesis, it is a macroscopic theory which describes the most probable behaviour of the system consisting of a large number of particles. For that reason, the equations of fluid mechanics must describe averaged motion of the points of the body along the smoothed worldlines.

Contrary to usual averaging procedures, according to which every dynamical variable is independently considered as the sum of smooth and non-smooth components (see, e.g., [24]), we regard all fluid mechanics fields as closely connected with each other. Hence, a random disturbance in any of these fields gives rise to disturbances in others. Thus, it is required to choose one of the hydromechanical fields which will be thought of as independent and will be written as the sum of smooth and pulse components. The rest of the fields will then be considered as smooth (i.e. non-random) if they are computed using the smooth field of an independent variable. Evidently, the independent variable should be one of the primary characteristics of the medium. The 4-velocity of the point of the body naturally can be taken as such.

Indeed, the starting concepts in our fluid mechanics considerations are the body $\mathcal{B}$, its image $\mathcal{B}^4$ in space of events, and measures of both of these objects. They are mass of the body $\mu(\mathcal{B})$, which induce mass $m(\mathcal{B}_t)$ of the worldtube cut (see [19] for details) and its volume $\omega(\mathcal{B}_t)$. Measure m is time-independent, whereas measure ω is in general a time-dependent function. It can easily be shown that the velocity $\mathrm{d}_t\omega$ is defined by the vector field $\vec{v}$ tangent to the worldlines, namely, if $\mathcal{B}_0 = \mathcal{B}_t$ when $t = 0$ is a reference configuration of the body and $\Omega = \omega(\mathcal{B}_0)$, then

$$\omega(\mathcal{B}_t) = \int_\Omega J \, \mathrm{d}\omega.$$

The volume change velocity is then as follows,

$$\mathrm{d}_t\omega = \int_\Omega \mathrm{d}_t J \, \mathrm{d}\omega = \int_\omega \frac{1}{J} \mathrm{d}_t J \, \mathrm{d}\omega = \int_\omega \operatorname{div} \vec{v} \, \mathrm{d}\omega \tag{10}$$

and this gives a differential relation which is valid at an arbitrary point of $\mathcal{B}^4$

$$\frac{1}{J} \mathrm{d}_t J = \operatorname{div} \vec{v}. \tag{11}$$

Note that formulation of the motion scaling problem makes sense just at this very step of development of the theory, although neither the mass density nor the pressure has yet been introduced.

Thus, the only time-dependent measure which may be regarded as containing a stochastic component is the volume of the configuration of the body. According to (10) evolution of this measure is defined by the velocity vector field. The velocity $\vec{v}$ and Jacobian J are those local characteristics of the moving fluid which must be treated as primary in contrast to the mass density, the pressure, and so forth. Since the theory of fluid motion is further developed in terms of velocity, it is convenient to use the velocity as the quantity which undergoes averaging.

The averaging operator (denoted by the overbar) must be a continuous linear projector mapping set of non-smoothed objects onto a set of averaged objects [24]. We represent the initial non-averaged worldline $\lambda(t)$ as the sum $\bar{\lambda}(t)+\lambda'(t)$ of a smoothed curve $\bar{\lambda}(t)$ and a pulse curve $\lambda'(t)$, then the velocity vector $\vec{v}(t)$ will divide into two parts $\overline{\vec{v}}(t) + \vec{v}'(t)$. The first one is tangent to a smoothed worldline and the second, to a pulse curve. Vector $\vec{v}$ is understood as the sum of two elements of tangent vector space.

3.2. Metrics

Metrics in terms of velocity of mean motion are defined such that the length of the non-averaged velocity vector is equal to unity

$$|\vec{v}|^2 = \mathsf{g}_{\alpha\beta} v^\alpha v^\beta = 1.$$

Applying the averaging procedure we get

$$\overline{|\vec{v}|^2} = \overline{|\bar{\vec{v}} + \vec{v}'|^2} = \mathsf{g}_{\alpha\beta}(\bar{v}^\alpha \bar{v}^\beta + \overline{v'^\alpha v'^\beta}).$$

The quantity

$$\bar{k} = \tfrac{1}{2}\rho|\bar{\vec{v}}|^2 = \tfrac{1}{2}\rho \mathsf{g}_{\alpha\beta}\bar{v}^\alpha \bar{v}^\beta \tag{12}$$

is called the *kinetic energy density of the mean motion*, and the quantity

$$e = \tfrac{1}{2}\rho\overline{|\vec{v}'|^2} = \tfrac{1}{2}\rho \mathsf{g}_{\alpha\beta}\overline{v'^\alpha v'^\beta} \tag{13}$$

is called the *internal energy density.* Besides

$$2\frac{k+e}{\rho} = 1. \tag{14}$$

In the case of orthogonal basis, $\mathsf{g} = g_0\mathsf{I}$. Equation (14) using (12) and (13) is rewritten as

$$g_0\delta_{\alpha\beta}\bar{v}^\alpha\bar{v}^\beta + 2\varepsilon = 1$$

where ε denotes specific internal energy density, $\varepsilon = \frac{e}{\rho}$. Metric coefficient g_0 is then defined by

$$g_0 = \frac{1-2\varepsilon}{\sum_\alpha(\bar{v}^\alpha)^2}. \tag{15}$$

3.3. Total energy conservation law

As has been mentioned in the subsection 2.3, the quantity $\frac{1}{2}m(\mathcal{B}_t)$ may be treated as the kinetic energy of the worldtube cut. Introduction of the averaging procedure divides this quantity into two parts, namely $K(\mathcal{B}_t)$, kinetic energy of the mean motion (note that we use the same symbol in a new meaning), and $E(\mathcal{B}_t)$, internal energy of the worldtube cut

$$\tfrac{1}{2}m(\mathcal{B}_t) = K(\mathcal{B}_t) + E(\mathcal{B}_t) \qquad K, E > 0.$$

The left-hand side is then called *total energy* of the configuration $\mathcal{B}_t$ of the body $\mathcal{B}$. Derivatives of K and E with respect to measure ω are densities k and e introduced above. In the thermodynamic case, equation (4) does not change and equation (8) turns into the total energy $k+e$ conservation law

$$d_t((k+e)J) = 0 \tag{16}$$

or

$$\operatorname{div}(k+e)\vec{v} = \rho d_t\frac{k+e}{\rho} = 0. \tag{17}$$

4. Viscosity and heat conduction

Heat conductivity and viscous momentum transfer by tradition are considered separately. In both cases the notion of surface interaction is employed. This approach for causal description of the viscous momentum transfer has been used in [19]. Another approach is developed here.

4.1. Averaging the equations

One of the main ideas underlying fluid mechanics consists of scale separation of motion. Large-scale motions are described explicitly, whereas small-scale motions are described parametrically. The momentum and energy exchange between both groups of motions are formulated in terms of viscosity and conductivity. The equations for large-scale evolution of the fluid are derived by averaging the source differential conservation laws of mass and energy

$$\overline{\operatorname{div}(\rho\vec{v})} = 0 \tag{18}$$

$$\overline{\operatorname{div}(k\vec{v})} = 0. \tag{19}$$

Due to continuity and linearity of the averaging operator the first linear equation gives a continuity equation of mean motion

$$\overline{\operatorname{div}(\rho\vec{v})} = \operatorname{div}\overline{\rho\vec{v}} = \operatorname{div}(\rho\bar{\vec{v}}) = 0. \tag{20}$$

Contrary to (18), equation (19) is nonlinear with respect to $\vec{v}$, which prevents derivation of a relation as simple as (20). To derive an equation for the kinetic energy, we shall average the kinetic energy flux density first, since $\overline{\operatorname{div}(k\vec{v})} = \operatorname{div}\overline{k\vec{v}}$. So,

$$\begin{aligned}\overline{k\vec{v}} &= \tfrac{1}{2}\rho \mathsf{g}_{\alpha\beta}\overline{v^\alpha v^\beta \vec{v}} \\ &= \tfrac{1}{2}\rho \mathsf{g}_{\alpha\beta}(\bar{v}^\alpha\bar{v}^\beta\bar{\vec{v}} + \overline{v'^\alpha v'^\beta}\bar{\vec{v}} + \bar{v}^\alpha\overline{v'^\beta\vec{v}'} + \overline{v'^\alpha\vec{v}'}\bar{v}^\beta + \overline{v'^\alpha v'^\beta\vec{v}'}) \\ &= \tfrac{1}{2}\rho(\bar{v}_\alpha\bar{v}^\alpha\bar{\vec{v}} + \overline{v'_\alpha v'^\alpha}\bar{\vec{v}} + 2\bar{v}^\alpha\overline{v'_\alpha\vec{v}'} + \overline{v'_\alpha v'^\alpha\vec{v}'}) \\ &= (\bar{k}+e)\bar{\vec{v}} + \rho\bar{v}^\alpha\overline{v'_\alpha\vec{v}'} + \rho\overline{\varepsilon'\vec{v}'}.\end{aligned}$$

Here ε' denotes $\frac{1}{2}v'_\alpha v'^\alpha$ (cf formula (13)). Calculating the divergence we find

$$\operatorname{div}\overline{k\vec{v}} = (\overline{kv^\gamma})_{;\gamma} = ((\bar{k}+e)v^\gamma)_{;\gamma} + (\rho\bar{v}^\alpha\overline{v'_\alpha v'^\gamma})_{;\gamma} + (\rho\overline{\varepsilon' v'^\gamma})_{;\gamma} = 0 \tag{21}$$

where $(\cdot)_{;\gamma}$ denotes the covariant derivative in the direction of the basis vector $\vec{e}_\gamma$. Covariances in (21) which describe the energy interchange of different types of motion are often expressed via gradients of the corresponding quantities (see, e.g., [24]), namely

$$\begin{aligned}\overline{v'_\alpha v'^\gamma} &= \mathsf{g}^{\gamma\beta}\overline{v'_\alpha v'_\beta} = \chi\mathsf{g}^{\gamma\beta}(\bar{v}_{\alpha;\beta} + \bar{v}_{\beta;\alpha}) \\ \overline{\varepsilon' v'^\gamma} &= \mathsf{g}^{\gamma\beta}\overline{\varepsilon' v'_\beta} = \theta\mathsf{g}^{\gamma\beta}\varepsilon_{,\beta}.\end{aligned}$$

Here $(\cdot)_{,\beta}$ stands for partial derivative with respect to x^β and χ and θ are coefficients of proportionality. The expression for velocity covariance includes a symmetric part of the velocity gradient only since the tensor $\overline{v'_\alpha v'_\beta}$ is symmetric. With respect to the orthogonal basis we obtain

$$\rho\overline{v'_\alpha v'^\gamma} = -\mu(\bar{v}_{\alpha;\beta} + \bar{v}_{\beta;\alpha})\delta^{\gamma\beta} \qquad \mu = -g_0^{-1}\chi\rho \tag{22}$$

$$\rho\overline{\varepsilon' v'^\gamma} = -\kappa\rho\varepsilon_{,\beta}\delta^{\gamma\beta} \qquad \kappa = -g_0^{-1}\theta. \tag{23}$$

Using these expressions equation (21) may now be written as follows (from now on we omit the overbar notation regarding the corresponding quantities as averaged):

$$((k+e)v^\gamma)_{;\gamma} + (2\rho\chi v^\alpha\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma} + (\rho\theta\mathsf{g}^{\gamma\beta}\varepsilon_{,\beta})_{;\gamma} = 0. \tag{24}$$

The indices in brackets stand for symmetrization, i.e. $2v_{(\alpha;\beta)} = v_{\alpha;\beta} + v_{\beta;\alpha}$. Due to (17) the first term in (24) is equal to zero and hence

$$(2\rho\chi v^\alpha\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma} = -(\rho\theta\mathsf{g}^{\gamma\beta}\varepsilon_{,\beta})_{;\gamma}. \tag{25}$$

The left-hand side describes part of the energy of mean motion which transforms into internal energy. The right-hand side is the source of internal energy connected with redistribution of internal energy.

4.2. Balance equations

4.2.1. Kinetic energy balance equation. Now, removing in (24) the first parenthesis, calculating the derivative in the second parenthesis and grouping the equation terms we find

$$\underbrace{\rho\,\mathrm{d}_t\frac{k}{\rho}+v^\alpha(2\rho\chi\,\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma}}_{=d_t\pi}+\underbrace{\rho\,\mathrm{d}_t\varepsilon+2\rho\chi\,v^\alpha_{;\gamma}\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)}+(\rho\theta\mathsf{g}^{\gamma\beta}\varepsilon_{,\beta})_{;\gamma}}_{=-d_t\pi}=0. \tag{26}$$

Denoting both groups by $\pm\mathrm{d}_t\pi$ we obtain the balance equations for different kinds of energy. Thus the first group gives the kinetic energy density balance equation

$$\rho\,\mathrm{d}_t\frac{k}{\rho}=\mathrm{d}_t\pi-v^\alpha(2\rho\chi\,\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma} \tag{27}$$

$$=\mathrm{d}_t\pi+2\rho\chi\,v^\alpha_{;\gamma}\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)}-(2\rho\chi\,v^\alpha\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma} \tag{28}$$

$$=\mathrm{d}_t\pi+2\rho\chi\,v^\alpha_{;\gamma}\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)}-\left(\rho\chi\,\mathsf{g}^{\gamma\beta}\left(\frac{k}{\rho}\right)_{,\beta}\right)_{;\gamma}-(\rho\chi\,\mathrm{d}_t v^\gamma)_{;\gamma}. \tag{29}$$

The terms on the right-hand side describe the sources and sinks of kinetic energy due to: compression/expansion ($\mathrm{d}_t\pi$), dissipation of kinetic energy $2\rho\chi\,v^\alpha_{;\gamma}\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)}$ and homogenization of disturbances $(2\rho\chi\,v^\alpha\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma}$ in the kinetic energy distribution.

4.2.2. Momentum balance equation. The momentum balance equation can be easily derived using the kinetic energy balance equation, namely

$$\begin{aligned}0&=\rho\,\mathrm{d}_t\frac{k}{\rho}-\mathrm{d}_t\pi+v^\alpha(2\rho\chi\,\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma}\\&=v^\alpha(\rho v^\gamma v_{\alpha;\gamma}-\pi_{,\alpha}+(2\rho\chi\,\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)})_{;\gamma})\\&=v_\alpha\big(\rho\,\mathrm{d}_t v^\alpha-\mathsf{g}^{\alpha\beta}\pi_{,\beta}+\big(\rho\chi\big(\mathsf{g}^{\gamma\beta}v^\alpha_{;\beta}+\mathsf{g}^{\alpha\beta}v^\gamma_{;\beta}\big)\big)_{;\gamma}\big).\end{aligned}$$

To within the vector orthogonal to $\vec{v}$ this gives

$$\rho\,\mathrm{d}_t v^\alpha=\mathsf{g}^{\alpha\beta}\pi_{,\beta}-\big(\rho\chi\big(\mathsf{g}^{\gamma\beta}v^\alpha_{;\beta}+\mathsf{g}^{\alpha\beta}v^\gamma_{;\beta}\big)\big)_{;\gamma}. \tag{30}$$

In the case of orthogonal basis when $\mathsf{g}^{\alpha\beta}=g_0^{-1}\delta^{\alpha\beta}$ we find

$$\rho\,\mathrm{d}_t v^\alpha=-\delta^{\alpha\beta}p_{,\beta}+\big(\mu\big(\delta^{\gamma\beta}v^\alpha_{;\beta}+\delta^{\alpha\beta}v^\gamma_{;\beta}\big)\big)_{;\gamma}. \tag{31}$$

Here $p\equiv-\pi g_0^{-1}$ denotes the *pressure* and μ is known as the *dynamic viscosity coefficient*. Equation (30) as well as equation (31) have been previously obtained in [19] (cf equation (13) and the stress tensor definition (20) therein).

4.2.3. Internal energy balance equation. The second group of terms in (26) gives an internal energy density balance equation

$$\rho\,\mathrm{d}_t\varepsilon=-\mathrm{d}_t\pi-2\rho\chi\,v^\alpha_{;\gamma}\mathsf{g}^{\gamma\beta}v_{(\alpha;\beta)}-(\rho\theta\mathsf{g}^{\gamma\beta}\varepsilon_{,\beta})_{;\gamma}. \tag{32}$$

The right-hand side also contains terms which describe the sources of internal energy. The first two terms coincide with the corresponding terms in the kinetic energy balance equation, but have opposite sign. The loss/growth of kinetic energy leads to an increase/decrease of internal energy. The last term describes redistribution of internal energy due to lessening of

the gradients. Using the metrics and definitions (22) and (23) this equation may be written in a more familiar notation

$$\rho\, \mathrm{d}_t \epsilon = \mathrm{d}_t p + 2\mu v^{\alpha}_{;\gamma} \mathsf{g}^{\gamma\beta} v_{(\alpha;\beta)} + (\rho\kappa\epsilon_{,\beta})_{;\gamma}\delta^{\gamma\beta}. \tag{33}$$

Here ϵ stands for $g_0^{-1}\varepsilon$. If mean motion is absent, equation (32) reads

$$\rho\partial_t \varepsilon = -\partial_t \pi - (\rho\theta \mathsf{g}^{\gamma\beta}\varepsilon_{,\beta})_{;\gamma}$$

and describes evolution of the internal energy field due to homogenization of disturbances. In the case of orthogonal basis this equation simplifies to

$$\rho\partial_t \varepsilon = -\partial_t \pi + (\rho\kappa\varepsilon_{,\beta})_{;\gamma}\delta^{\gamma\beta}.$$

The quantity κ is called the *coefficient of thermal diffusivity*.

4.2.4. Heat equation. Now it is possible to derive an equation for temperature T which is defined via the relation $\mathrm{d}\epsilon = c_p\, \mathrm{d}T$. Here c_p is the *heat capacity* of the medium *at constant pressure*. Substituting this definition into equation (33) we obtain

$$\rho c_p\, \mathrm{d}_t T = \mathrm{d}_t p + 2\mu v^{\alpha}_{;\gamma} \mathsf{g}^{\gamma\beta} v_{(\alpha;\beta)} + (\rho\kappa c_p T_{,\beta})_{;\gamma}\delta^{\gamma\beta}$$

or in the absence of mean motion

$$\rho c_p \partial_t T = \partial_t p + (\lambda T_{,\beta})_{;\gamma}\delta^{\gamma\beta}. \tag{34}$$

Equation (34) is called the *heat conduction equation*. The coefficient $\lambda = \kappa\rho c_p$ is known as the *coefficient of thermal conductivity*.

Equations (33) and (34) are of second order and are hyperbolic since they both contain the second derivative with respect to time, which has the proper sign due to imaginarity of the zeroth coordinate.

5. Diffusion

Contrary to viscosity and heat conduction, diffusion is the transport of mass and thus it is a current. However, there are some specific features which prevent one from using the standard means of description.

5.1. Peculiarities and problems

Peculiarities

1. The body $\mathcal{B}$ is regarded to be the union $\mathcal{B} = \bigcup_E \mathcal{B}^E$ of $E = 1, \ldots, n$ components $\mathcal{B}^E$. Configurations of these components will be denoted as $\mathcal{B}^E_t \subset \mathcal{B}_t$. In place of the single measure $m(\mathcal{B}_t)$, mass of the body, new n measures $m_E = m\big(\mathcal{B}^E_t\big)$ are introduced, each of which is interpreted as the mass of the corresponding component. Besides, due to additivity of the measure
$$m(\mathcal{B}_t) = \sum_E m_E. \tag{35}$$
2. When interaction between the components is purely mechanical, the motion of each component may be considered separately as the motion of a conserved quantity [1]
$$d_t m_E = 0 \quad \Rightarrow \quad d_t m = 0. \tag{36}$$
3. In the general case, each measure is connected with its own transport, i.e. its own current. This means that in place of a single velocity field, n vector fields are defined, one vector field per component. Each vector field gives rise to congruence of the worldlines, the union of which coincides with the worldtube of the body.

Problems and approaches

1. Several congruences within a single worldtube manifest themselves in the fact that each point of the body is connected with numerous worldlines. Thus, an important property of deformation, bijectivity, is lost. Each point moves in different directions at the same time. However, there is no contradiction here since we are speaking about the points of the continuum, and various mappings $\mathcal{W} \to \mathbb{R}^4$, not molecules or particles of the matter. Nevertheless, the velocity becomes an inconvenient way for description of the motion. We further use another method: convective transport is substituted for diffusive transport.
2. For a single fluid the metrics has been defined in terms of convective velocity, i.e. vector tangent to the worldline of the point of the body. For the mixtures we must either define different metrics for each component or prefer one vector field to all others and use it in the definition of the metrics. We follow this way.

5.2. Mass density

We define mass density ρ_E of the Eth component of the mixture (the so-called *partial density*) according to (1) as a derivative of the measure m_E with respect to the volume of configuration $\omega_E = \omega\big(\mathcal{B}_t^E\big)$ of component

$$m_E = \int_{\omega_E} \rho_E \, \mathrm{d}\omega. \tag{37}$$

However, those parts of $\mathcal{B}_t$ which do not contain points of the component have zero partial density of the corresponding component and, hence

$$m_E = \int_{\omega} \rho_E \, \mathrm{d}\omega. \tag{38}$$

Using (35) the mass of the body in terms of the volume of the current configuration is written as

$$m = \int_{\omega(\mathcal{B}_t)} \rho \, \mathrm{d}\omega = \sum_E \int_{\omega(\mathcal{B}_t)} \rho_E \, \mathrm{d}\omega. \tag{39}$$

Since the latter equality holds for an arbitrary mixture, one obtains

$$\rho = \sum_E \rho_E \tag{40}$$

i.e. mass density of the mixture is equal to the sum of partial densities.

Mass m in terms of volume $\Omega = \omega(\mathcal{B}_0)$ of the reference configuration $\mathcal{B}_0$, reads

$$m = \int_{\Omega} \rho J \, \mathrm{d}\omega. \tag{41}$$

Here J is the Jacobian of the Lagrangian to Euler coordinate transformation of points of the mixture which rectifies the worldtube $\mathcal{B}^4$ of the body (the mixture as a whole). For mass m_E in its turn a similar reasoning is valid, i.e.

$$m_E = \int_{\Omega_E} \rho_E J_E \, \mathrm{d}\omega = \int_{\Omega} \rho_E J_E \, \mathrm{d}\omega \tag{42}$$

where $\Omega_E = \omega\big(\mathcal{B}_0^E\big)$, and J_E is the Jacobian of the coordinate transformation of the points of the Eth component which rectifies its worldtube (i.e. worldtube of $\mathcal{B}^E$). The latter equality in (42) holds due to the above-stated reasons (see (37) and (38)).

5.3. Congruence

Each body $\mathcal{B}$ corresponds to unique worldtube $\mathcal{B}^4$. However, each worldtube corresponds to infinitely many congruences of worldlines. To make use of this non-uniqueness, we bind each component of the mixture with a separate congruence. The vector field of the component induces a congruence of the worldlines $\hat{\mathcal{B}}^4_E$, which is a submanifold of worldtube $\mathcal{B}^4$. We may supplement $\hat{\mathcal{B}}^4_E$ up to $\mathcal{B}^4$ using an arbitrary congruence of worldlines, and get the congruence $\mathcal{B}^4_E$, i.e. the worldtube of the Eth component. Thus, worldtubes of each of these congruences, which are in general different, coincide with the worldtube of the body $\mathcal{B}^4$.

5.3.1. Congruence of the mixture (single fluid congruence). Let us define the centre of the mass x_* of the configuration $\mathcal{B}_t$ as follows:

$$x_*(\mathcal{B}_t) = \frac{\int_{\omega(\mathcal{B}_t)} \rho x \,\mathrm{d}\omega}{\int_{\omega(\mathcal{B}_t)} \rho \,\mathrm{d}\omega} = \frac{1}{m(\mathcal{B}_t)} \int_{\omega(\mathcal{B}_t)} \rho x \,\mathrm{d}\omega = x_*(t, \mathcal{B}).$$

Let infinite sequence $\mathcal{B}, \mathcal{P}^1, \mathcal{P}^2, \ldots$, which consists of parts of the body $\mathcal{B} \supset \mathcal{P}^1 \supset \mathcal{P}^2 \supset \ldots$ be such that intersection of all of its terms $\cap_k \mathcal{P}^k = \mathcal{P}$ is equal to the point of the body $\mathcal{P} \in \mathcal{B}$, i.e. there is only one point common to all of these parts. The worldline of this point is defined as follows:

$$x(t, \mathcal{P}) = \lim_{k\to\infty} \frac{1}{m\big(\mathcal{P}^k_t\big)} \int_{\omega(\mathcal{P}^k_t)} \rho x \,\mathrm{d}\omega. \tag{43}$$

Here $\mathcal{P}^k_t \subset \mathcal{B}_t$ is the image of $\mathcal{P}^k$ in the space of events at time t. Similarly we construct worldlines for all other points of the body. These worldlines in total form the *congruence* of the worldtube of the body *which corresponds to the density field* ρ.

5.3.2. Congruence of the component of the mixture. Let the body be a mixture of n components each of which corresponds to partial mass density field ρ_E on the worldtube. Every component corresponds to its own collection of the worldlines which are defined by a relation similar to (43)

$$x_E(t, \mathcal{P}) = \lim_{k\to\infty} \frac{1}{m_E\big(\mathcal{P}^k_t\big)} \int_{\omega_E(\mathcal{P}^k_t)} \rho_E x \,\mathrm{d}\omega = \lim_k \frac{1}{m^k_E} \int_{\omega(\mathcal{P}^k_t)} \rho_E x \,\mathrm{d}\omega. \tag{44}$$

The latter equality holds since $\rho_E = 0$ outside $\omega_E\big(\mathcal{P}^k_t\big)$. Distribution of partial density ρ_E differs in general from ρ and congruences of the worldlines connected with relations (43) and (44) do not coincide being in total one and the same worldtube $\mathcal{B}^4$ in both cases. When $\rho_E = 0$, the congruence is arbitrary since the velocity vector field does not take part in the transport of mass.

5.3.3. Connection between congruences of the mixture and its components. Connection of the mass of the mixture and its density field with masses and densities of its components is known (relations (35) and (40)). Now it is possible to find a connection between different congruences of a given worldtube. Definition (43) gives

$$\begin{aligned} x(t, \mathcal{P}) &= \lim_k \frac{1}{m^k} \int_{\omega^k} \rho x \,\mathrm{d}\omega = \lim_k \frac{1}{m^k} \int_{\omega^k} \left(\sum_E \rho_E\right) x \,\mathrm{d}\omega \\ &= \sum_E \lim_k \frac{1}{m^k} \int_{\omega^k} \rho_E x \,\mathrm{d}\omega = \sum_E \lim_k \frac{m^k_E}{m^k} \frac{1}{m^k_E} \int_{\omega^k} \rho_E x \,\mathrm{d}\omega \\ &= \sum_E C_E x_E(t, \mathcal{P}). \end{aligned}$$

Here $m^k = m\big(\mathcal{P}_t^k\big)$, $m_E^k = m_E\big(\mathcal{P}_t^k\big)$, $\omega^k = \omega\big(\mathcal{P}_t^k\big)$, and $C_E = \lim_k \frac{m_E^k}{m^k}$ denotes a derivative of the measure m_E with respect to the measure m, which is called the *concentration* of the Eth component of the mixture. Thus, congruence of the mixture is the weighted-mean congruence of its components.

5.4. Vector fields on $\mathcal{B}^4$

5.4.1. Velocity vectors and the tangent vector spaces. Set $\mathcal{T}$ of the vectors tangent to the worldtube at some point does not coincide with set $\mathcal{V}$ of the permissible velocity vectors at the same point. Each velocity vector has a non-zero zeroth component $v^0 = \mathrm{i}c \neq 0$ with respect to an arbitrary basis. Due to this feature of the velocity vectors it is impossible to introduce a structure of the vector space on the set $\mathcal{V}$, because the zero vector is not an element of this set. However, this set may be treated as an affine space, and each element $\vec{v} \in \mathcal{V}$ may be considered as the sum $\vec{v} = \vec{\tau} + \vec{u}$. Here $\vec{\tau} \in \mathcal{T}$ is a fixed tangent vector and $\vec{u} \in \mathcal{U}$, where $\mathcal{U}$ is a subspace of $\mathcal{T}$. It is convenient to choose $\vec{\tau} = (\mathrm{i}c, 0, 0, 0)$. Then the set $\mathcal{U}$ will consist of tangent vectors with zeroth component equal to 0, i.e. vectors tangent to the configuration of the body.

Classical fluid mechanics defines the vector spaces using the component-wise addition and multiplication by number. In the four-dimensional context operations are defined on the elements of $\mathcal{U}$, which becomes the vector space since the corresponding axioms are satisfied.

Note that the sum of two 4-velocity vectors considered as the elements of the affine space $\mathcal{V}$ is not defined whereas the remainder of the same vectors regarded as elements of the tangent vector space $\mathcal{T}$ is defined and belongs to $\mathcal{U} \subset \mathcal{T}$.

5.4.2. Non-uniqueness of vector field on $\mathcal{B}^4$. By virtue of conservation law (36) for the mixture and its components, differentiating (41) and (42) with respect to t leads to the following equalities which are valid along the corresponding worldlines

$$\mathrm{d}_t(\rho J) = 0 \qquad \mathrm{d}_t(\rho_E J_E) = 0 \qquad \forall E. \tag{45}$$

Taking into account that $\mathrm{d}_t J = \operatorname{div} \vec{v}$ and $\mathrm{d}_t J_E = \operatorname{div} \vec{v}_E$ we obtain

$$\operatorname{div} \rho\vec{v} = 0 \qquad \operatorname{div} \rho_E \vec{v}_E = 0. \tag{46}$$

Here $\vec{v} = \vec{v}(\mathcal{P}_t)$ and $\vec{v}_E = \vec{v}_E(\mathcal{P}_t)$ are velocity vectors tangent to worldlines $x(t, \mathcal{P})$ and $x_E(t, \mathcal{P})$ which pass through the point $\mathcal{P}_t$. The same is valid for any other point of the configuration of the body. This means that a non-unique vector field is defined on a worldtube and there exist different congruences which make one and the same worldtube $\mathcal{B}^4$. As has been noted in subsection 5.4, vectors $\vec{v}$ and $\vec{v}_E$ belong to the affine space of permissible velocity vectors $\mathcal{V}$, and thus are equal to $\vec{v} = \vec{\tau} + \vec{w}$ and $\vec{v}_E = \vec{\tau} + \vec{w}_E$. In turn, the vectors $\vec{w} = (0, w^1, w^2, w^3)$ and $\vec{w}_E = \big(0, w_E^1, w_E^2, w_E^3\big)$ are elements of the tangent vector space $\mathcal{U}$. Differentiating (43) and taking into account (45) one finds

$$\vec{w}(\mathcal{P}_t) = \mathrm{d}_t x(t, \mathcal{P}) = \lim_k \frac{1}{m^k} \int_{\omega^k} \rho\vec{w}\, \mathrm{d}\omega = \lim_k \frac{1}{m^k} \int_{\omega^k} \left(\sum_E \rho_E \right) \vec{w}\, \mathrm{d}\omega$$

$$= \sum_E \lim_k \frac{m_E^k}{m^k} \frac{1}{m_E^k} \int_{\omega^k} \rho_E \vec{w}\, \mathrm{d}\omega = \sum_E C_E \vec{w}_E(\mathcal{P}_t)$$

where

$$\vec{w}_E(\mathcal{P}_t) = \lim_k \frac{1}{m_E^k} \int_{\omega^k} \rho_E \vec{w}\, \mathrm{d}\omega.$$

At points where the partial density is equal to zero the input of the velocity component is also zero, due to $C_E = 0$. Hence, the velocity field (as well as congruence) in these domains may be arbitrary.

The following equalities give a connection between the mass density flux of the mixture $\rho\vec{v}$ and fluxes of mixture components $\rho_E\vec{v}_E$

$$\begin{aligned}\rho\vec{v} &= \rho(\vec{\tau}+\vec{w}) = \rho\left(\vec{\tau}+\sum_E C_E\vec{w}_E\right) = \rho\vec{\tau}+\sum_E \vec{w}_E\rho C_E \\ &= \rho\vec{\tau}+\sum_E \vec{w}_E \lim_k \frac{m^k}{\omega^k}\frac{m^k_E}{m^k} = \rho\vec{\tau}+\sum_E \vec{w}_E \lim_k \frac{m^k_E}{\omega^k} \\ &= \rho\vec{\tau}+\sum_E \vec{w}_E\rho_E = \sum_E \rho_E\vec{\tau}+\sum_E \vec{w}_E\rho_E \\ &= \sum_E \rho_E(\vec{\tau}+\vec{w}_E) = \sum_E \rho_E\vec{v}_E.\end{aligned}$$

Here we make use of equalities (37) and (38). Besides, the following relations are valid

$$\rho_E = C_E\rho \quad \Rightarrow \quad \sum_E C_E = 1.$$

Thus, we must either be able to distinguish between different vector fields or choose an alternative way to describe the motion, a way which does not explicitly include the velocity vector. We shall try to utilize this possibility. Using the continuity equation we exclude the velocity where possible or construct the scalar div $\vec{v}$ which defines the scalar field on $\mathcal{B}^4$ and is frequently assumed to be constant.

5.5. Metrics

Considering the motion of components as independent requires either introduction of n different metrics since the latter are defined through the velocity field or preferring one such field to all others and definition of the metric tensor via this very field. It is quite natural to choose the velocity field $\vec{v}$ as such. Definition of metric tensor g is then based on the condition

$$|\vec{v}|^2 = \mathsf{g}_{\alpha\beta}v^\alpha v^\beta = 1-2\varepsilon$$

where ε is specific internal energy density. If the basis is orthogonal, then $\mathsf{g}_{\alpha\beta} = g_0\delta_{\alpha\beta}$ and metric coefficient g_0 is defined by relation

$$g_0 = \frac{1-2\varepsilon}{\sum_\alpha v^\alpha v^\alpha}. \tag{47}$$

Formally, this is just the relation (15). However, vector $\vec{v}$ has another meaning which here is velocity of the mixture as a whole.

5.6. Derivation of diffusion equation

The mass of diffusing substance (component of the mixture) conserves, and hence the continuity equation for the Eth component is the starting point of our reasoning

$$\operatorname{div}\rho_E\vec{v}_E = 0. \tag{48}$$

Let $\vec{u}_E \equiv \vec{v}_E - \vec{v}$ be the velocity of the Eth component relative to motion of the mixture as a whole. Equation (48) is then written as follows:

$$\operatorname{div}\rho_E(\vec{v}+\vec{u}_E) = (\rho\vec{v},\nabla C_E) + \operatorname{div}\rho_E\vec{u}_E = \rho d_t C_E + \operatorname{div}\rho_E\vec{u}_E = 0. \tag{49}$$

The first term on the right-hand side describes transport of the component via the velocity field of mean motion. Individual velocities are contained in the second term only. Let us try to substitute them for gradients of the corresponding densities. For the rest mixture, or in a co-moving system of coordinates, equation (49) reads

$$\operatorname{div} \rho_E \vec{u}_E = 0. \tag{50}$$

Calculating the divergence in (50) one obtains

$$u_E^\alpha \frac{1}{\rho_E} \nabla_\alpha \rho_E = -\operatorname{div} \vec{u}_E.$$

To within the term orthogonal to $\vec{u}_E$, we assume

$$\frac{1}{\rho_E} \nabla_\alpha \rho_E = -\frac{\operatorname{div} \vec{u}_E}{|\vec{u}_E|^2} u_{E\alpha}. \tag{51}$$

Denoting $\zeta_E = \frac{|\vec{u}_E|^2}{\operatorname{div} \vec{u}_E}$ one finds

$$\rho_E \vec{u}_E = -\zeta_E \mathsf{g}^{-1} \nabla \rho_E$$

and further

$$\operatorname{div} \rho_E \vec{u}_E = -\operatorname{div} (\zeta_E \mathsf{g}^{-1} \nabla \rho_E).$$

Thus, a change in the partial mass density, which is not connected with transport by mean motion, we have written in terms of the gradient of the corresponding quantity. Equation (48) now reads

$$\operatorname{div} \rho_E \vec{v}_E = \rho \, \mathrm{d}_t C_E - \operatorname{div} (\zeta_E \mathsf{g}^{-1} \nabla \rho_E) = 0. \tag{52}$$

We further change the second term such that the whole equation is written in terms of concentration C_E. Since

$$\nabla \rho_E = \nabla (C_E \rho) = \rho \nabla C_E + C_E \nabla \rho$$

we get

$$\operatorname{div} (\zeta_E \mathsf{g}^{-1} \nabla \rho_E) = \operatorname{div} (\zeta_E \mathsf{g}^{-1} \rho \nabla C_E) + \operatorname{div} (\zeta_E \mathsf{g}^{-1} C_E \nabla \rho). \tag{53}$$

The second term in (53) may be written as follows:

$$\operatorname{div} \left(\zeta_E \mathsf{g}^{-1} \rho_E \frac{1}{\rho} \nabla \rho \right). \tag{54}$$

We express the gradient of mean density similarly to (51). The continuity equation leads to $\frac{1}{\rho} \mathrm{d}_t \rho = -\operatorname{div} \vec{v}$, and to within the term orthogonal to vector $\vec{v}$ we obtain

$$\frac{1}{\rho} \nabla_\alpha \rho = -\frac{\operatorname{div} \vec{v}}{|\vec{v}|^2} v_\alpha$$

or, denoting $\zeta = \frac{|\vec{v}|^2}{\operatorname{div} \vec{v}}$

$$\frac{1}{\rho} \nabla_\alpha \rho = -\frac{1}{\zeta} v_\alpha.$$

Substituting this expression into (54), we find

$$\begin{aligned} \operatorname{div} \left(\zeta_E \mathsf{g}^{-1} \rho_E \frac{1}{\rho} \nabla \rho \right) &= -\operatorname{div} (\rho_E \zeta_E \zeta^{-1} \vec{v}) = -\operatorname{div} (C_E \zeta_E \zeta^{-1} \rho \vec{v}) \\ &= -\rho \, \mathrm{d}_t (C_E \zeta_E \zeta^{-1}). \end{aligned}$$

Now equation (52) looks like

$$\operatorname{div} \rho_E \vec{v}_E = \rho \, \mathrm{d}_t C_E + \rho \, \mathrm{d}_t (C_E \zeta_E \zeta^{-1}) - \operatorname{div} (\zeta_E \mathsf{g}^{-1} \rho \nabla C_E) = 0$$

or

$$\rho \, \mathrm{d}_t C_E (1 + \zeta_E \zeta^{-1}) = \operatorname{div} (\zeta_E \mathsf{g}^{-1} \rho \nabla C_E). \tag{55}$$

In addition to concentrations of the components this equation contains only one velocity vector, the velocity of the mixture as a whole. Since the values of the ratio $\zeta_E \zeta^{-1}$ are usually negligibly small, equation (55) may be written in a more familiar form

$$\rho \, \mathrm{d}_t C_E = \operatorname{div} (\xi_E \rho \nabla C_E). \tag{56}$$

Besides, it is assumed here that $\mathsf{g} = g_0 \mathsf{I}$. Coefficient ξ_E denotes the ratio $\xi_E = \zeta_E g_0^{-1}$, which is called the *coefficient of diffusion* of the Eth component. Equation (56) in turn is called the *diffusion equation* of the corresponding component in a moving continuum. This equation is hyperbolic since the operator on the right-hand side contains a second-order derivative with respect to time and thus retains the causal relationship of events.

6. Discussion

The approach suggested in [19] allows derivation of hyperbolic equations for heat and momentum transfer. An analysis of the diffusion phenomenon shows that this approach is applicable in this case also. The hyperbolic diffusion equation derived describes equalization of disturbances in the n component mixture. Thus, all kinds of transport phenomena may be described using equations which do not contradict the causality principle. In what follows we make some concluding remarks.

1. Despite the formal similarity of the equations which describe both groups of phenomena (viscosity–conductivity and diffusion) their nature in the framework of continuum mechanics is fundamentally different. The first group describes the momentum/energy transport due to high-frequency velocity fluctuations which we are unable to describe explicitly and use a parametric description. The second group is connected with transport of the matter and thus should be described explicitly. This transport does not connect with the high-frequency fluctuations (since all of them are already included in internal energy) and it should not be confused with other transport phenomena. The problem consists of non-uniqueness of the vector field on manifold $\mathcal{B}^4$. The vector fields are generated by different components of the mixture. However, all of these fields are usually close to each other and the description method utilizes this feature. Since diffusion is a material flow, in the case of a single fluid both variants of description are possible. Indeed, along with an ordinary problem of description of the fluid flow, the self-diffusion phenomenon is also considered.
2. The above-written considerations concerning the nature of transport phenomena do not guarantee causal stipulation of the corresponding description. The signal velocity used for observation of fluid motion may be regarded as such a guarantee. A similar fact is well known in the numerical analysis. An explicit method applied to a parabolic equation actually substitutes the latter for a hyperbolic one. This occurs due to the following reason. Any explicit method allows computation of a solution on a current time level using the known solution from the previous levels. In this case, the velocity specified by the method appears in a numerical analogue of the problem (usually this is discretization steps ratio) as the upper limit for velocities permitted by the algorithm. Note that this limit is not absolute, i.e. higher velocities are not forbidden, simply they will be interpreted in

the wrong way (see [19] for details). This feature of the explicit numerical procedures is well known as *conditional stability*.

3. Unlike most of known methods of deriving hyperbolic equations for transport phenomena, the current issue deals with the continuum only without resorting to molecules and/or particles of the matter. We tried to show that classical non-causal description of the phenomena is connected not with substitution of the set of molecules for the continuum and refusal to describe the detailed interaction of the microscopic particles, but with the assumption of infinite velocity of signal propagation. Rejection of this assumption allowed derivation in all three cases of causal equations which model the corresponding transport processes. This idea is verified by the study of the classical limit of the equations obtained. Namely, at $c \to \infty$ we get the usual equations of motion of viscous fluid, as well as equations of heat and mass transfer.

Acknowledgments

I would like to thank my colleagues who patiently discussed these problems with me and especially Dr A Safrai whose arguments and criticism were extremely useful. E V Popov read the manuscript and made many valuable remarks.

References

[1] Batchelor G 1970 *An Introduction to Fluid Dynamics* (Cambridge: Cambridge University Press)
[2] Jackson H E and Walker C T 1971 Thermal conductivity, second sound, and phonon–phonon interactions in NaF *Phys. Rev.* B **3** 1428–39
[3] Narayanamurti V and Dynes R C 1972 Observation of second sound in bismuth *Phys. Rev. Lett.* **28** 1461–5
[4] Peshkov V 1944 Second sound in helium II *J. Phys. USSR* **8** 381–6
[5] Yang H Q 1991 Non-Fourier effect on heat conduction during welding *Int. J. Heat Mass Transfer* **34** 2921–4
[6] Tang D W and Araki N 1996 Non-Fourier heat conduction in a finite medium under periodic surface thermal disturbance *Int. J. Heat Mass Transfer* **39** 1585–90
[7] Vedavarz A, Kumar S and Moallemi M K 1994 Significance of non-Fourier heat waves in continuum *J. Heat Transfer* **116** 116–224
[8] Richtmyer R D 1978 *Principles of Advanced Mathematical Physics* vol 1 (New York: Springer)
[9] Fichera G 1922 Is the Fourier theory of heat propagation paradoxical? *Rend. Circ. Mat. Palermo* **41** 5–28
[10] Villaggio P 2001 The apparent propagation velocity of a wave *Rend. Mat. Acc. Lincei, Mec. Solidi* **12** 191–7
[11] Cattaneo C 1958 A form of heat conduction equation which eliminates the paradox of instantaneous propagation *C. R. Acad. Sci. Paris* **247** 431–3
[12] Chen G 2001 Ballistic-diffusive heat conduction equation *Phys. Rev. Lett.* **86** 2297–300
[13] Joseph D D and Preziosi L 1989 Heat waves *Rev. Mod. Phys.* **61** 41–73
[14] Joseph D D and Preziosi L 1990 Addendum to the paper: heat waves *Rev. Mod. Phys.* **62** 375–91
[15] Jou D, Casa-Vázquez J and Lebon G 1988 Extended irreversible thermodynamics *Rep. Prog. Phys.* **51** 1105–79
[16] Jou D, Casa-Vázquez J and Lebon G 1999 Extended irreversible thermodynamics revisited 1988–1998 *Rep. Prog. Phys.* **62** 1035–114
[17] Barletta A and Zanchini E 1997 Hyperbolic heat conduction and local equilibrium: a second law analysis *Int. J. Heat Mass Transfer* **40** 1007–16
[18] Depireux N and Lebon G 2001 An extended thermodynamics modeling of non-Fickian diffusion *J. Non-Newtonian Fluid Mech.* **96** 105–17
[19] Belevich M 2003 Causal description of non-relativistic dissipative fluid motion *Acta Mech.* **161** 65–80
[20] Truesdell C A 1977 *First Course in Rational Continuum Mechanics* (New York: Academic)
[21] Pauli W 1958 *Theory of Relativity* (New York: Pergamon)
[22] Arnold V I 1973 *Ordinary Differential Equations* (Cambridge, MA: MIT Press)
[23] Kolmogorov A N and Fomin S V 1975 *Introductory Real Analysis* (New York: Dover)
[24] Monin A S and Yaglom A M 1971 *Statistical Fluid Mechanics* vol 1 (Cambridge, MA: MIT Press)